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Symmetrical reconfiguration of tensegrity structures

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Abstract

In this article we first present a mathematical model which describes the nonlinear dynamics of tensegrity structures. For certain tensegrity structures a particular class of motions, coined symmetrical motions, is defined. The corresponding equations of motion are derived and the conditions under which symmetrical motions occur are established. Reconfiguration procedures through symmetrical motions are proposed and examples are given. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Tensegrity structures; Dynamics; Deployment; Reconfiguration

1. Introduction

Tensegrity structures represent a special class of space structures composed of a set of *soft* members—which can carry only tension forces (e.g. tendons)—and a set of *hard* ones (for example bars). These structures are characterized by *prestressability*, which is the property to maintain an equilibrium shape with all soft members in tension and in the absence of external forces or torques. These structures integrity is guaranteed by the soft members in tension, hence their denomination, *tensegrity*, an acronym of *tension integrity*. Tensegrity structures are capable of large displacement, belonging to the class of flexible structures. They offer excellent opportunities for physically integrated structure and controller design, since the elastic as well as the rigid components can carry both sensing and actuating functions. A perspective view of a tensegrity structure composed of 24 tendons and six bars is given in Fig. 1.

The origins of tensegrity structures go back to 1921 (see Sadao, 1996). Inspired by Kenneth Snelson sculptures created in 1948 (see Snelson, 1996), Buckminster Fuller patented a class of tendon-bars structures which he called tensegrity structures. Later work by engineers and scientists (Calladine, 1978, 1982; Connelly, 1980; Pellegrino and Calladine, 1986; Calladine and Pellegrino, 1991; Motro, 1992; Ingber, 1993; Connelly and Whiteley, 1996; Skelton and Sultan, 1997; Sultan et al., 2000) generalized the term tensegrity structures.

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Nomenclature

b	length of the base and top triangles sides
$\hat{b}_{1,2,3}$	inertial reference frame unit vectors
d	common value of the coefficients of friction at all joints
d_j	the coefficient of friction at joint j
k_j	stiffness of the jth tendon
$k_{S,V,D}$	stiffness of the saddle (S), vertical (V), diagonal (D) tendons respectively
l	length of a bar
l_j	length of the jth tendon
l_{0_j}	rest length of the jth tendon
m	mass of a bar
q	vector of generalized coordinates
$\dot{\vec{r}}^n$	velocity of the nth rigid body mass center
t	time
$\hat{t}_{1,2,3}$	top reference frame unit vectors
D	length of a diagonal tendon in a symmetrical configuration
D_0	rest length of a diagonal tendon
E	number of tendons
F	vector of external forces
F_i	external force acting on the $\hat{t}_i$ axis
$\vec{F}^n$	resultant nonconservative force applied to rigid body n
J	transversal moment of inertia of a bar
K	kinetic energy
L	the Lagrangian
L_j	elongation of the jth tendon
$M(q)$	inertia matrix
M_t	mass of the top
M_i	external torque acting on the $\hat{t}_i$ axis
$\vec{M}^n$	resultant nonconservative torque applied to rigid body n
$\vec{M}_{f_j}$	friction torque at joint j
N	number of degrees of freedom
Q	vector of nonconservative generalized forces
R	number of rigid bodies
S	length of a saddle tendon in a symmetrical configuration
S_0	rest length of a saddle tendon
V_0	rest length of a vertical tendon
X, Y, Z	Cartesian inertial coordinates of the mass center of the top
$T(q)$	vector of tensions in the tendons
T_j	tension in the jth tendon
T_D	tension in a diagonal tendon in a symmetrical configuration
T_S	tension in a saddle tendon in a symmetrical configuration
T_V	tension in a vertical tendon in a symmetrical configuration
U	potential energy
V	length of a vertical tendon in a symmetrical configuration
Z_i	initial height

Z_f	final height
α	azimuth of bar $A_{11}B_{11}$ in a symmetrical configuration
α_i	initial azimuth
α_f	final azimuth
α_{ij}	azimuth of bar $A_{ij}B_{ij}$
δ	declination of a bar in a symmetrical configuration
δ_i	initial declination
δ_f	final declination
δ_{ij}	declination of bar $A_{ij}B_{ij}$
ψ, ϕ, θ	Euler angles of the top reference frame
τ	reconfiguration time
$\vec{\omega}^n$	angular velocity of the nth rigid body

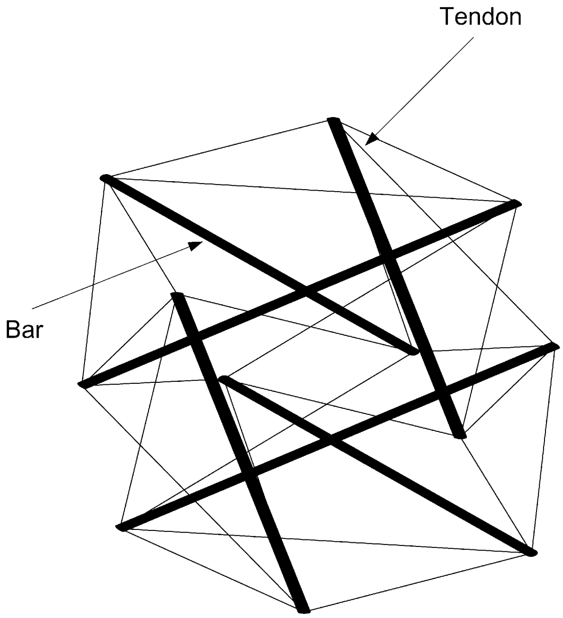

Fig. 1. A tensegrity structure.

Fuller (1975) and Pugh (1976) pioneered tensegrity structures research, but their work was confined to geometrical investigations. Structural mechanics was later involved as the theoretical framework for the analysis and design of these structures and research in tensegrity structures turned into a systematic one. Calladine (1978, 1982), Motro et al. (1986), Hanaor (1988), Pellegrino (1990) made important contributions toward further knowledge of the statics of these structures. Kebiche et al. (1999) presented interesting results of numerical nonlinear static analysis of tensegrity structures. Sultan et al. (2001) formulated the complete prestressability problem—which consists in finding equilibrium configurations with all tensile elements in tension when no external forces or torque act on the structure—and reported the discovery of analytical solutions of this problem for several classes of tensegrity structures.

Research in tensegrity structures dynamics is less developed than its statics counterpart. Previous investigations were limited to numerical and experimental analysis of particular tensegrity configurations. Motro et al. (1986) presented experimental results of linear dynamics for a tensegrity structure composed of three bars and nine tendons. The experiment was aimed at determining the dynamic characteristics of the structure by harmonic excitation of a node and measurement of the responses of the other nodes. Furuya (1992) used finite element programs to analyze the vibrational characteristics of some tensegrity structures and concluded that the modal frequencies increase as the pretension increases. Oppenheim and Williams (2001a) proved that, for a tensegrity structure composed of three bars and six tendons and with linear kinetic damping in the tendons, the rate of decay of vibrations is much slower than might be expected. However when linear kinetic friction of the angular motion between structural members in contact was introduced, an exponential rate of decay was obtained. In another article by Oppenheim and Williams (2001b) these remarks were reinforced, leading to the conclusion that friction in the rotational joints of the structure is a more important source of damping than the material damping in the tendons. Murakami (2001) used the Lagrangian and Eulerian approaches to the equations of motion derivation for a large class of structures and applied them to some tensegrity structures for numerical simulation and modal analysis.

Active control design studies for tensegrity structures have been reported by Skelton and Sultan (1997), Sultan and Skelton (1997, 1998a, 1998b), Sultan (1999), Sultan et al. (1999, 2000), and Djouadi et al. (1998). These articles showed that tensegrity structures are excellent candidates for smart structures since the control systems can be easily embedded in the structures; for example some of the tendons can act as actuators and some as sensors, providing the basic components of a control system.

Tensegrity structures also aroused the interest of the bio-medical community: they have been proposed to explain how various types of cells (e.g. nerve cells, smooth muscles, etc.) resist shape distortion (Ingber, 1993, 1998). Results of static numerical analysis using a tensegrity structure to model a cell's static properties which were in agreement with biological experimental measurements have been reported (see Stamenovic et al., 1996; Coughlin and Stamenovic, 1997).

In this article we first present the nonlinear equations of motion for certain tensegrity structures. Next, for particular tensegrity structures, a class of motions, coined symmetrical motions, is defined and the conditions under which these motions exist, as well as the corresponding equations of motion, are established. These symmetrical motions are then used in tackling the important problem of reconfiguration in tensegrity structures. Tendon control reconfiguration procedures through symmetrical motions are proposed. The reconfiguration takes place in a finite, prescribed, time. Examples of these reconfiguration procedures are given.

2. Nonlinear equations of motion

An important advantage of flexible tensegrity structures over classical flexible structures is that their dynamics can be described accurately enough by *ordinary differential equations* rather than *partial differential equations*. This is so because tensegrity structures flexibility is achieved through special design techniques which combine elements that can be considered, to a good approximation, *massless elastic* members (e.g. tendons) or *rigid* bodies. Under very general modeling assumptions this results, as we shall see in the following, in mathematical models composed of *finite* sets of *ordinary differential equations*.

As is well known, ordinary differential equations are much easier to deal with numerically as well as analytically. Moreover, modern control system design theory heavily relies on *state space* representation of the system's dynamics. Ordinary differential equations are readily put in state space form (see Skelton, 1988), whereas for partial differential equations the situation is different; the *separation of variables* method is applied in some cases to get an infinite set of ordinary differential equations and a set of partial differential equations with boundary values. Usually, for control design, only a finite set of ordinary differential equations is retained. Thus, *qualitative* as well as *quantitative* alteration of the original mathematical model is performed through this process.

Next we derive a mathematical model of tensegrity structures dynamics which consists of ordinary differential equations.

Consider a tensegrity structure composed of E *elastic* and *massless* tendons and R *rigid* bodies. We assume that all constraints on the system are *holonomic*, *scleronomic*, and *bilateral*. The external constraint forces are *workless*, which means that they do no work through a virtual displacement consistent with the geometric constraints. We neglect the forces exerted on the structure by other force fields (for example the gravitational field).

Let $q_j, j = 1, \ldots, N$, be a set of *independent* (also called *Lagrange*) generalized coordinates which describe the motion of the system with respect to an inertial reference frame and let

$$q = [q_1 \; q_2 \; \ldots \; q_N]^T \tag{1}$$

be the vector of generalized coordinates. The application of the Lagrangian methodology to derive the nonlinear equations of motion requires the derivation of the kinetic and potential energies and of the *nonconservative* generalized forces.

Since the tendons are massless the kinetic energy is given by the rigid bodies and it is a quadratic form of the generalized velocities:

$$K = \frac{1}{2}\dot{q}^T M(q)\dot{q}, \tag{2}$$

where $M(q)$ is the inertia matrix.

The potential energy is due to the E tendons, being given by

$$U = \sum_{j=1}^{E} \int_0^{L_j} T_j \, \mathrm{d}l_j. \tag{3}$$

Here L_j is the elongation of the jth tendon, T_j is its tension (considered positive if the tendon is in tension and zero otherwise), and the differential element $\mathrm{d}l_j$ is given by

$$\mathrm{d}l_j = \sum_{n=1}^{N} \frac{\partial l_j}{\partial q_n} \, \mathrm{d}q_n, \tag{4}$$

where l_j is the length of the jth tendon. The potential energy becomes

$$U = \sum_{j=1}^{E} \int_0^{L_j} T_j \sum_{n=1}^{N} \frac{\partial l_j}{\partial q_n} \, \mathrm{d}q_n. \tag{5}$$

We assume that the system is also acted upon by *nonconservative* forces and torques. The corresponding nonconservative generalized forces can be expressed as shown by Skelton (1988):

$$Q_j = \sum_{n=1}^{R} \left(\vec{F}^n \cdot \frac{\partial \dot{\vec{r}}^n}{\partial \dot{q}_j} + \vec{M}^n \cdot \frac{\partial \vec{\omega}^n}{\partial \dot{q}_j} \right), \quad j = 1, \ldots, N. \tag{6}$$

Here Q_j is the nonconservative generalized force associated with the jth generalized coordinate, $\vec{F}^n$ and $\vec{M}^n$ are the resultant nonconservative force and torque, respectively, applied to rigid body n, $\dot{\vec{r}}^n$ and $\vec{\omega}^n$ are the velocity of the center of mass and the angular velocity of the nth rigid body, respectively.

For a holonomic system whose configuration is described by N independent generalized coordinates q_j, $j = 1, \ldots, N$, Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j, \quad j = 1, \ldots, N, \tag{7}$$

where t is the time and $L = K - U$ is the Lagrangian of the system.

Applied to a tensegrity structure, Lagrange equations yield

$$M(q)\ddot{q} + c(q, \dot{q}) + A(q)T(q) = Q, \tag{8}$$

where

- $c(q, \dot{q})$ is a vector of quadratic functions in $\dot{q}$, whose components can be expressed as

$$c_i = \sum_{j=1}^{N} \sum_{n=1}^{N} \left(\frac{\partial M_{ij}}{\partial q_n} - \frac{1}{2}\frac{\partial M_{jn}}{\partial q_i}\right)\dot{q}_j\dot{q}_n, \quad i = 1, \ldots, N; \tag{9}$$

- $A(q)T(q)$ is the vector of elastic generalized forces where $A[n, j] = \partial l_j/\partial q_n$, $n = 1, \ldots, N$, $j = 1, \ldots, E$, and $T(q)$ is the vector of tensions in the tendons;
- $Q = [Q_1 \; Q_2 \; \cdots \; Q_N]^T$ is the vector of nonconservative generalized forces.

Eqs. (8), which describes tensegrity structures nonlinear dynamics, represent a finite set of second order ordinary differential equations.

A particular case of interest is when the nonconservative forces and torques acting on the structure can be separated in the following two types. The first type is that of *linear kinetic friction* forces and torques at the joints of the structure and *linear kinetic damping* forces in the tendons (a linear kinetic friction force/torque is proportional to the relative linear/angular velocity between the members in contact, whereas a linear kinetic damping force is proportional to the time derivative of the tendon's elongation), and the second type is that of external—not friction or damping—forces and torques applied to the rigid bodies. Using Eqs. (6) it is easy to see that in this case the generalized forces are linear in the generalized velocities and in the external forces and torques, leading to the following equations of motion:

$$M(q)\ddot{q} + c(q, \dot{q}) + A(q)T(q) + C(q)\dot{q} + H(q)F = 0, \tag{10}$$

where $C(q)$ and $H(q)$ are matrices of appropriate dimensions whereas F is the vector of external forces and torques.

3. Two stage SVD tensegrity structures

In the following we shall focus on a certain class of structures, coined two stage SVD tensegrity structures. These structures present clear practical interest for industrial applications. They have been previously investigated for control design (Skelton and Sultan, 1997), integrated structure and control system design (Sultan and Skelton, 1997), sensors design (Sultan and Skelton, 1998a), deployment (Sultan and Skelton, 1998b), and prestressability (Sultan et al., 2001).

A two stage SVD tensegrity structure is composed of six bars, a rigid top ($B_{12}B_{22}B_{32}$), a rigid base ($A_{11}A_{21}A_{31}$), and 18 tendons, as shown in Fig. 2. A stage contains bars with the same second index; for example bars $A_{11}B_{11}$, $A_{21}B_{21}$, $A_{31}B_{31}$ determine the first stage. The acronym "SVD" comes from the following notation we introduce for the tendons: tendons $B_{i1}A_{j2}$ will be called *saddle* tendons, $A_{j1}B_{i1}$ and $A_{j2}B_{i2}$ *vertical* tendons, and $A_{j1}A_{i2}$ and $B_{j1}B_{i2}$ *diagonal* tendons respectively.

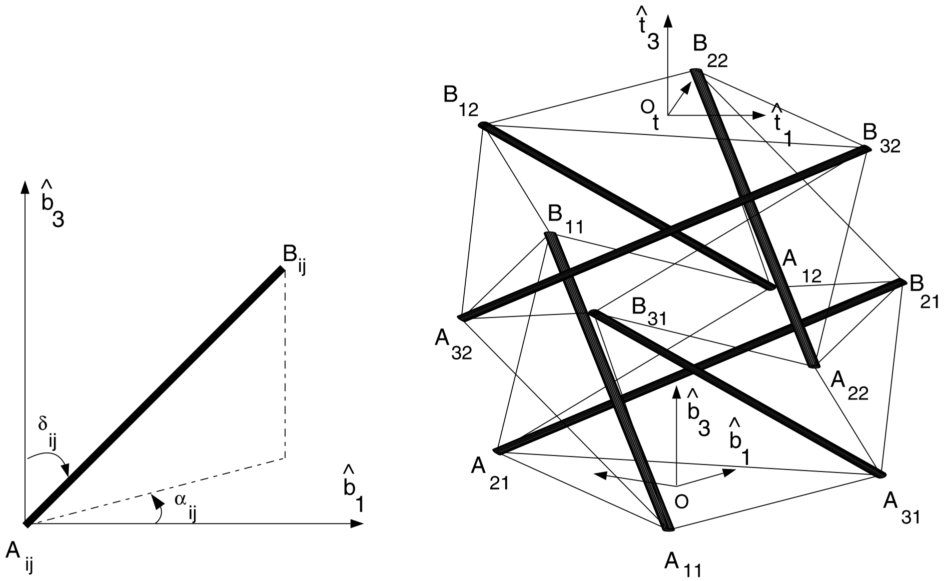

Fig. 2. Two stage SVD tensegrity structure.

The assumptions made for the mathematical modeling of two stage SVD tensegrity structures are: the tendons are *massless*, *not-damped* (not affected by damping), and *linear elastic*, the tension in tendon j being given by

$$T_j = k_j \frac{l_j - l_{0_j}}{l_{0_j}}, \tag{11}$$

where k_j is the stiffness of the jth tendon (defined here as the product between the cross-section and the longitudinal modulus of elasticity of the tendon), l_j is its length, and l_{0_j} is its rest–length. We also assume that the base and the top are *rigid* bodies, the bars are *rigid*, *axially symmetric*, and for each bar the rotational degree of freedom around the longitudinal axis of symmetry is neglected. We assume that an external force and an external torque act on the rigid top. Also friction torques, proportional to the relative angular velocity, act at the six joints between the base or the rigid top with the bars, being given by

$$\vec{M}_{f_j} = d_j\left(\vec{\omega}^{a} - \vec{\omega}^{b}\right), \quad j = 1, \ldots, 6, \tag{12}$$

where $\vec{M}_{f_j}$ is the friction torque at joint j, exerted by body "b" on body "a", due to the relative angular motion between bodies "a" and "b". The scalar $d_j \leqslant 0$ is the coefficient of friction at joint j, whereas $\vec{\omega}^{*}$ is the angular velocity of body *. We neglect the forces exerted on the structure by external force fields (e.g. gravitational). We remark that these assumptions are particular cases of the modeling assumptions made for the derivation of the general equations of motion of tensegrity structures. The equations of motion of two stage SVD tensegrity structures have the form given by Eqs. (10).

We consider that the base is fixed and we introduce the *inertial* reference frame, $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_3$, as an orthonormal dextral set of vectors, whose origin coincides with the geometric center of triangle $A_{11}A_{21}A_{31}$. Axis $\hat{b}_3$ is orthogonal to $A_{11}A_{21}A_{31}$ pointing upward, while $\hat{b}_1$ is parallel to $A_{11}A_{31}$, pointing toward A_{31}. We introduce another orthonormal dextral reference frame, $\hat{t}_1$, $\hat{t}_2$, $\hat{t}_3$, called the top reference frame. Its origin is located at the geometric center of triangle $B_{12}B_{22}B_{32}$, O_t, axis $\hat{t}_3$ is orthogonal to $B_{12}B_{22}B_{32}$ pointing upward, while $\hat{t}_1$ is parallel to $B_{12}B_{32}$, pointing toward B_{32}. For simplicity it is assumed that the top reference frame is *central principal* for the top rigid body.

The 18 independent generalized coordinates used to describe the motion of this *holonomic* system, are: ψ, ϕ, θ, the Euler angles for a 3-1-2 sequence to characterize the inertial orientation of the top reference frame, X, Y, Z, the inertial Cartesian coordinates of the origin of the top reference frame, δ_{ij} and α_{ij}, the declination and the azimuth of the longitudinal axis of symmetry of bar $A_{ij}B_{ij}$, measured with respect to the inertial reference frame and defined as follows: δ_{ij} is the angle made by the vector $\overrightarrow{A_{ij}B_{ij}}$ with $\hat{b}_3$ and α_{ij} is the angle made by the projection of this vector onto plane ($\hat{b}_1$, $\hat{b}_2$) with $\hat{b}_1$ (see Fig. 2). Hence, the vector of generalized coordinates is

$$q = [\delta_{11}\ \alpha_{11}\ \delta_{21}\ \alpha_{21}\ \delta_{31}\ \alpha_{31}\ \delta_{12}\ \alpha_{12}\ \delta_{22}\ \alpha_{22}\ \delta_{32}\ \alpha_{32}\ \psi\ \phi\ \theta\ X\ Y\ Z]^T. \tag{13}$$

The nonlinear equations of motion are given by Eqs. (10):

$$M(q)\ddot{q} + c(q,\dot{q}) + A(q)T(q) + C(q)\dot{q} + H(q)F = 0. \tag{14}$$

Matrices $M(q)$, $A(q)$, $C(q)$, $H(q)$ have been derived by Sultan (1999). Here F is the vector of external forces and torques acting on the rigid top, given by

$$F = [M_1\ M_2\ M_3\ F_1\ F_2\ F_3]^T, \tag{15}$$

where M_i and F_i, $i = 1$–3, are the force and torque, respectively, acting on the rigid top along axis $\hat{t}_i$.

4. Symmetrical motions

We shall now investigate the conditions under which particular motions, described by simpler equations, are possible. We first assume that the two stage SVD tensegrity structure under investigation has the following properties.

- All bars are identical, of length l and mass m, and the top and base triangles are equal equilateral triangles with sides of length b.
- All saddle tendons are identical (of rest length S_0 and stiffness k_S); all vertical tendons are identical (of rest length V_0 and stiffness k_V); all diagonal tendons are identical (of rest length D_0 and stiffness k_D).
- The coefficients of friction at the six joints—between the base or the rigid top with the bars—are equal, $d_j = d$, $j = 1, \ldots, 6$.

We introduce a particular class of configurations called *symmetrical configurations* and defined as follows: all bars have the same declination, δ, the vertical projections of points A_{i2}, B_{i1}, $i = 1$–3, onto the base make a regular hexagon, planes $A_{11}A_{21}A_{31}$ and $A_{12}A_{22}A_{32}$ are parallel. These configurations can be parameterized by three numbers: the azimuth of bar $A_{11}B_{11}$, called α, the declination, δ, and the height of the structure, Z. The generalized coordinates corresponding to a symmetrical configuration are given by the following expressions:

$$\begin{aligned}&\psi = 300, \quad \phi = \theta = 0, \quad X = Y = 0, \quad Z = Z, \quad \alpha_{11} = \alpha, \quad \alpha_{21} = \alpha + 240,\\ &\alpha_{31} = \alpha + 120, \quad \alpha_{12} = \alpha + 120, \quad \alpha_{22} = \alpha, \quad \alpha_{32} = \alpha + 240, \qquad \delta_{ij} = \delta, \quad i = 1\text{–}3,\ j = 1,2.\end{aligned} \tag{16}$$

From physical considerations we impose the following restrictions. First $\alpha \in \{[0, 360) - 30\}$ since for $\alpha = 30$ some bars—for example $A_{11}B_{11}$, $A_{21}B_{21}$, $A_{31}B_{31}$—intersect. Second $0 < \delta < 90$ because for $\delta = 90$ the bars collide with the base or top and for $\delta = 0$ they are perpendicular to the base and top, a situation we do not

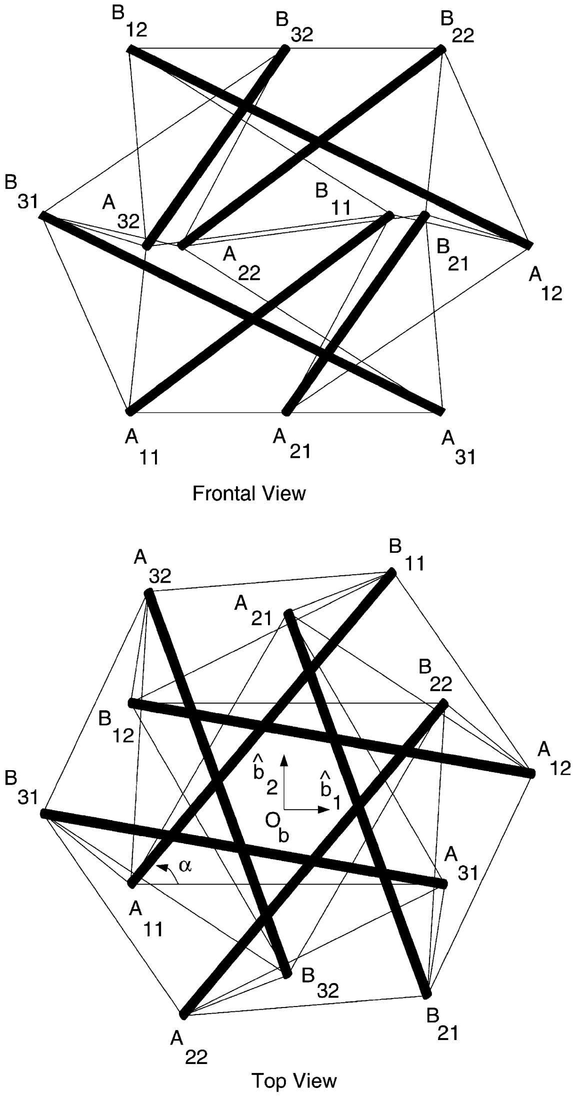

Fig. 3. Symmetrical configuration.

consider here. Lastly the height, Z, should be large enough such that the ends of the bars of the second stage labeled A_{i2}, $i = 1$–3, do not collide with the base, hence $Z > l\cos(\delta)$ (because for $Z = l\cos(\delta)$ impact occurs).

Top and frontal views of a two stage SVD tensegrity structure with this geometry are given in Fig. 3.

In a symmetrical configuration, the lengths of all of the saddle (S), all of the vertical (V), and all of the diagonal (D) tendons are the same, given by

$$S = \left\{ Z^2 + l^2 + 3l^2\cos^2(\delta) - 4lZ\cos(\delta) + \frac{b^2}{3} - \frac{2}{\sqrt{3}} lb\sin(\delta)\cos(\alpha - 30) \right\}^{1/2}, \tag{17}$$

$$V = \{ l^2 + b^2 - 2lb\sin(\delta)\sin(\alpha + 30) \}^{1/2}, \tag{18}$$

$$D = \left\{ l^2 + \frac{b^2}{3} + Z^2 - 2lZ\cos(\delta) - \frac{2}{\sqrt{3}} lb\sin(\delta)\sin(\alpha) \right\}^{1/2}, \quad (19)$$

respectively (see Sultan et al., 2001, for details on the derivation of S, V, D). The corresponding tensions are all equal; let their values be T_S, T_V, T_D respectively.

The particular class of motions we are interested in and which we shall call *symmetrical motions*, is characterized by the fact that throughout the motion the structure yields symmetrical configurations. The corresponding generalized coordinates have the values given by Eqs. (16).

Substitution of Eqs. (16) into the general equations of motion, Eqs. (14), leads to the following equations (see Sultan, 1999, for details):

$$F_1 = F_2 = M_1 = M_2 = 0, \quad (20)$$

$$\frac{d}{dt}(\dot{Z}\sin(\delta)) = 0, \quad (21)$$

$$\frac{d}{dt}\left(3\tilde{J}\sin^2(\delta)\dot{\alpha} + \frac{\sqrt{3}}{2} mlb \frac{d}{dt}(\sin(\delta)\cos(\alpha + 60))\right) = M_3, \quad (22)$$

$$\tilde{J}\ddot{\delta} - d\dot{\delta} + A_1^T T_r = 0, \quad (23)$$

$$\tilde{J}\sin^2(\delta)\ddot{\alpha} - d\dot{\alpha} + \tilde{J}\sin(2\delta)\dot{\alpha}\dot{\delta} + A_2^T T_r = 0, \quad (24)$$

$$\tilde{m}\sin(\delta)\ddot{\delta} + \widetilde{M}\ddot{Z} + \tilde{m}\cos(\delta)\dot{\delta}^2 + A_3^T T_r = F_3, \quad (25)$$

where

$$\tilde{J} = J + \frac{ml^2}{4}, \qquad \tilde{m} = \frac{3ml}{2}, \qquad \widetilde{M} = M_t + 3m. \quad (26)$$

Here A_1^T, A_2^T, A_3^T are the rows of the matrix A_r given in Appendix A, M_t is the mass of the rigid top, J is the transversal moment of inertia of a bar, and $T_r = [T_S\ T_V\ T_D]^T$. Additionally, throughout the motion, the tendons must be in tension, thus

$$T_S > 0, \qquad T_V > 0, \qquad T_D > 0. \quad (27)$$

We note that a *necessary* condition for symmetrical motions to exist is that the components of the force and the torque acting on the rigid top, except those along the $\hat{t}_3$ axis, must be zero ($F_1 = F_2 = M_1 = M_2 = 0$).

Another *necessary* condition for these motions to exist is given by Eq. (21) which says that the component of the rigid top velocity onto a plane perpendicular to a bar is constant throughout the motion. Consider now the case when the initial derivative of Z, let it be called $\dot{Z}_i$, is nonzero, implying $\dot{Z}_i\sin(\delta_i) \neq 0$ (since $0 < \delta_i < 90$). Then, from Eq. (21), we get

$$\dot{Z} = \frac{\dot{Z}_i\sin(\delta_i)}{\sin(\delta)} \Rightarrow |\dot{Z}| > |\dot{Z}_i\sin(\delta_i)| \quad (28)$$

which shows that the height of the structure would either continuously increase or continuously decrease (as $t \to \infty$, $Z \to -\infty$ or $Z \to +\infty$). Because of physical limitations this is possible only for a limited time (for example until some members of the structure break). If $\dot{Z}_i = 0$ then from Eq. (21) we get $Z =$ constant (since $\sin(\delta) = 0$ leads to $\delta = 180\ n$ with n an integer number, which is not an acceptable solution because $0 < \delta < 90$). Thus during such a symmetrical motion the height of the structure remains constant and the configuration of the structure changes only through changes in α and δ (the attitude of the bars). Hence, the only rigid bodies in motion are the bars.

We remark that, in general, symmetrical motions are described by five equations, Eqs. (21)–(25), whereas we have only three independent variables, α, δ, Z. This is a consequence of the fact that we imposed certain constraints, given by Eqs. (16), on the generalized coordinates. Because of this we cannot arbitrarily specify the initial conditions (namely the values and first order time derivatives of α, δ, and Z at the initial time) and the controls time histories (the external force, F_3, torque, M_3, and the tendons rest lengths, S_0, V_0, D_0). For example assume that we specify the initial conditions and the tendons rest-lengths time histories. Then, by the theorem of the existence and uniqueness solution of the initial conditions problem for ordinary differential equations, Eqs. (23)–(25) yield a unique solution for α, δ, and Z. Thus M_3 cannot be arbitrarily specified but has to satisfy Eq. (22). Moreover, this solution might violate condition (21). This discussion reveals an important problem for these motions, which consists in finding the initial conditions and the controls which result in symmetrical motions. In this article we shall address the most interesting problem (from a practical perspective) namely that of equilibrium initial conditions, when all the time derivatives are zero initially. This results in a very useful application of symmetrical motions called *symmetrical reconfiguration*.

4.1. Symmetrical equilibrium configurations

Symmetrical equilibrium configurations are characterized by Eqs. (20)–(25) in which all time derivatives are zero, thus

$$F_1 = F_2 = M_1 = M_2 = M_3 = 0, \tag{29}$$

and

$$A_r T_r = [0\ 0\ F_3]^T, \qquad T_S > 0, \qquad T_V > 0, \qquad T_D > 0. \tag{30}$$

If $F_3 = 0$ then the corresponding symmetrical equilibrium configurations are actually symmetrical prestressable configurations for which all saddle, vertical, and diagonal tendons tensions are respectively equal (see Sultan et al., 2001, for analytical solutions of conditions (30) in this case).

If $F_3 \neq 0$ then we can easily prove that, because α, δ, and Z must satisfy $\alpha \in \{[0, 360) - 30\}$, $0 < \delta < 90$, and $Z > l\cos(\delta)$, there are no solutions of $A_r T_r = [0\ 0\ F_3]^T$ for which $\det(A_r) = 0$. Indeed, such a solution occurs if the first two rows of A_r are linearly dependent, a condition which, after simple algebraic manipulations, leads to $Z = l\cos(\delta)$ which contradicts the condition $Z > l\cos(\delta)$ (for $Z = l\cos(\delta)$ the ends of the second stage bars labeled A_{i2}, $i = 1$–3, collide with the base and the ends of the first stage labeled B_{i1}, $i = 1$–3, collide with the top).

Hence A_r is nonsingular at a solution for which $F_3 \neq 0$ and $A_r T_r = [0\ 0\ F_3]^T$ can be easily solved for the tensions yielding

$$T_r = A_r^{-1}[0\ 0\ F_3]^T. \tag{31}$$

5. Symmetrical reconfiguration

Consider now the following scenario: the structure is in a symmetrical equilibrium configuration characterized by α_i, δ_i, and Z_i. We want to change this initial configuration to another symmetrical equilibrium configuration, characterized by α_f, δ_f, and Z_f through a symmetrical motion. Because the structure is in equilibrium initially, the symmetrical motion can take place only at constant height, thus $Z_i = Z_f = Z$. We call this process *symmetrical reconfiguration*. It can be used for packing the structure for transport or for modifying its mechanical properties (e.g. stiffness or dynamical characteristics) by changing its geometry. For example we can imagine an application in which a two stage SVD tensegrity structure is required to support a constant vertical load at a given height. Operating conditions require modification of the mechanical characteristics or of the internal geometry (bars attitude) of the structure without unloading it or changing its height. This can be done through a symmetrical reconfiguration.

If α and δ are prescribed as functions of time then we can solve for the necessary controls required to enforce a symmetrical reconfiguration, using Eqs. (22)–(25). For example we consider that the external force F_3 (which in this scenario is the load under which the structure operates) is given. Under the assumption that $A_r = [A_1\ A_2\ A_3]^T$ (which is a function of α and δ) is nonsingular we get

$$T_r = -A_r^{-1}e, \tag{32}$$

$$M_3 = \frac{\mathrm{d}}{\mathrm{d}t}\left(3\tilde{J}\sin^2(\delta)\dot{\alpha} + \frac{\sqrt{3}}{2}mlb\frac{\mathrm{d}}{\mathrm{d}t}(\sin(\delta)\cos(\alpha+60))\right), \tag{33}$$

where

$$e = \begin{bmatrix} \tilde{J}\ddot{\delta} - d\dot{\delta} \\ \tilde{J}\sin^2(\delta)\ddot{\alpha} - d\dot{\alpha} + \tilde{J}\sin(2\delta)\dot{\alpha}\dot{\delta} \\ \tilde{m}\sin(\delta)\ddot{\delta} + \tilde{m}\cos(\delta)\dot{\delta}^2 - F_3 \end{bmatrix}. \tag{34}$$

In addition the tensions should be positive: $T_S > 0$, $T_V > 0$, $T_D > 0$. We can see that, in this scenario, tendon control (that is control of the tendons tensions, T_r) *and* external torque control ($M_3 \neq 0$) are required.

Tendon control requires modification of the active lengths of the tendons. This task can be accomplished by motors attached, for example, at the end of the bars (or, if the bars are hollow, inside them). We consider that these motors work in the following way. For example the motor pulls a tendon and rolls it over a small wheel in such a way that its active length is shortened: the part of the tendon which is rolled over the motor wheel no longer contributes to the tendon tension. Hence this control procedure works as if the rest–length of the tendon would be shortened. Similarly, when the motor reverses the sense of rotation, a portion of the inactive tendon becomes active, carrying force; hence the rest–length of the tendon increases. We call this procedure of tendon control, *rest-length control.*

Taking into account the relations between the rest–lengths and the tensions under the assumption that the tendons are linear elastic, we can solve for the necessary rest–lengths which enforce the motion:

$$S_0 = \frac{k_S S}{T_S + k_S}, \qquad V_0 = \frac{k_V V}{T_V + k_V}, \qquad D_0 = \frac{k_D D}{T_D + k_D}, \tag{35}$$

where $T_S = T_{r1}$, $T_V = T_{r2}$, $T_D = T_{r3}$ and S, V, D are given by Eqs. (17)–(19).

We shall further determine the necessary characteristics of α and δ as functions of time connecting the initial and final configurations, $(\alpha_i, \delta_i, Z_i)$ and $(\alpha_f, \delta_f, Z_f)$ (where $Z_i = Z_f$), respectively. In the following we shall refer to these functions as $\alpha(t)$ and $\delta(t)$ respectively.

- First, we want the transition between the two symmetrical equilibrium configurations to take place in a *prescribed finite time*, τ.
- Second, since the terminal points are equilibrium configurations, α and δ must be constant before and after the transition, which is assumed to take place for $t \in [0, \tau]$. Thus:

$$\alpha(t) = \begin{cases} \alpha_i & \text{if } t \leqslant 0 \\ \alpha_f & \text{if } t \geqslant \tau \end{cases}, \qquad \delta(t) = \begin{cases} \delta_i & \text{if } t \leqslant 0 \\ \delta_f & \text{if } t \geqslant \tau. \end{cases} \tag{36}$$

- Third, practical considerations force us to consider only continuous time controls. Consequently the vector of tendons tensions, T_r, must be continuous in time. Hence, $e(t) = A_r T_r$ must be continuous in time, and thus $\alpha(t)$ and $\delta(t)$ must be functions of (at least) class $C^2(\mathbb{R})$. Thus, $\dot{\alpha}$, $\ddot{\alpha}$, $\dot{\delta}$, and $\ddot{\delta}$ should have zero values at the terminal points of the transition:

$$\dot{\alpha}(0) = \ddot{\alpha}(0) = \dot{\delta}(0) = \ddot{\delta}(0) = \dot{\alpha}(\tau) = \ddot{\alpha}(\tau) = \dot{\delta}(\tau) = \ddot{\delta}(\tau) = 0. \tag{37}$$

The simplest piecewise polynomials which satisfy the above conditions are:

$$\alpha(t) = \begin{cases} \alpha_{\mathrm{i}} & \text{if } t \leqslant 0 \\ \alpha_t & \text{if } 0 < t < \tau, \\ \alpha_{\mathrm{f}} & \text{if } t \geqslant \tau \end{cases} \tag{38}$$

where

$$\alpha_t = \alpha_{\mathrm{i}} + \frac{30}{\tau^5}(\alpha_{\mathrm{f}} - \alpha_{\mathrm{i}})\left(\frac{t^5}{30} - \frac{t^4}{6}(t-\tau) + \frac{t^3}{3}(t-\tau)^2\right), \tag{39}$$

$$\delta(t) = \begin{cases} \delta_{\mathrm{i}} & \text{if } t \leqslant 0 \\ \delta_t & \text{if } 0 < t < \tau, \\ \delta_{\mathrm{f}} & \text{if } t \geqslant \tau \end{cases} \tag{40}$$

where

$$\delta_t = \delta_{\mathrm{i}} + \frac{30}{\tau^5}(\delta_{\mathrm{f}} - \delta_{\mathrm{i}})\left(\frac{t^5}{30} - \frac{t^4}{6}(t-\tau) + \frac{t^3}{3}(t-\tau)^2\right). \tag{41}$$

The set of functions which satisfy the above conditions is very rich and other examples can be easily constructed. For example we can multiply α_t and/or δ_t by $(1 + t^2(t-\tau)^2)^n$ where n is a natural number, or by any other function $g(t)$ of class $C^2([0,\tau])$ such that $g(0) = g(\tau) = 1$ and $\dot{g}(0) = \ddot{g}(0) = \dot{g}(\tau) = \ddot{g}(\tau) = 0$ and the resulting $\alpha(t)$ and/or $\delta(t)$ will satisfy the required conditions.

Lastly, the condition that A_r is nonsingular deserves some discussion. The corresponding condition, det $(A_r) \neq 0$, reduces to

$$Z^2 u - Z\left(3lu + \frac{b}{\sqrt{3}}\right)\cos(\delta) + \left(3lu + \frac{\sqrt{3}}{2}b\right) l \cos^2(\delta) \neq 0, \tag{42}$$

where $u = \sin(\delta)\cos(\alpha + 30)$ (see Sultan et al., 2001, for details). The functions $\alpha(t)$ and $\delta(t)$ must satisfy this condition. Once $\alpha(t)$ and $\delta(t)$ have been chosen one easy way to verify if condition (42) is satisfied is through discretization by gridding the time interval $[0,\tau]$ and checking condition (42) at the nodes of the grid. We also remark that condition (42), being linear in $\cos(\alpha + 30)$, can be easily solved for α, providing another simple way to check if $\det(A_r) \neq 0$.

5.1. Example

Consider a two stage tensegrity structure characterized by the following parameters:

$$\begin{aligned} &l = 1\ \text{m}, \quad b = 0.67\ \text{m}, \quad J = 1\ \text{kg}\,\text{m}^2, \quad m = 1\ \text{kg}, \quad M_t = 1\ \text{kg}, \quad d = -1, \\ &k_{\mathrm{S}} = k_{\mathrm{V}} = k_{\mathrm{D}} = 1000\ \text{N}. \end{aligned} \tag{43}$$

The structure is acted upon by a constant force $F_3 = -50$ N. The corresponding equilibrium symmetrical configurations are characterized by conditions (30).

Solutions of conditions (30) can be found as follows. First we choose α, δ, and Z such that $\det(A_r) \neq 0$ (that is α, δ, and Z verify condition (42) and $\alpha \in \{[0, 360) - 30\},\ 0 < \delta < 90,\ Z > l\cos(\delta)$) and we compute the tensions using Eq. (31). If these are positive, then a feasible solution of the problem has been found, otherwise we have to modify α, δ, or Z.

We next consider the following two feasible solutions:

$$\alpha_i = 55^\circ, \qquad \delta_i = 40^\circ, \qquad Z_i = 1.11 \text{ m}, \qquad T_{r_i} = [26.11\ 1.85\ 18.38]^T \text{ N}, \tag{44}$$

and

$$\alpha_f = 65^\circ, \qquad \delta_f = 35^\circ, \qquad Z_f = Z_i = 1.11 \text{ m}, \qquad T_{r_f} = [46.97\ 12.27\ 43]^T \text{ N}. \tag{45}$$

The structure can evolve through a symmetrical motion from the initial configuration, characterized by $(\alpha_i, \delta_i, Z_i)$, to the final configuration, characterized by $(\alpha_f, \delta_f, Z_f)$. For example, for $\tau = 5$ s, the corresponding $\alpha(t)$ and $\delta(t)$ given by Eqs. (38)–(41), are plotted in Fig. 4. The necessary rest–lengths which

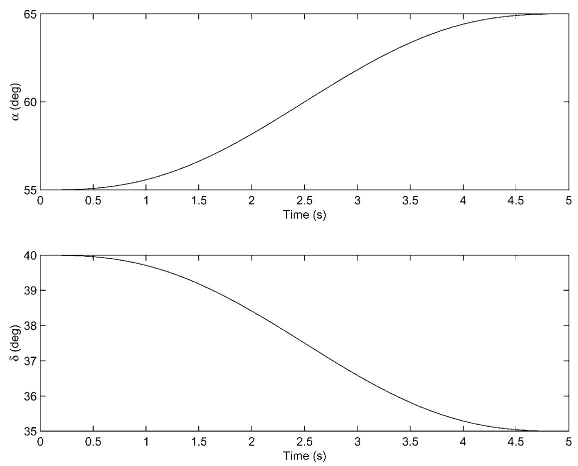

Fig. 4. α and δ time histories for symmetrical reconfiguration.

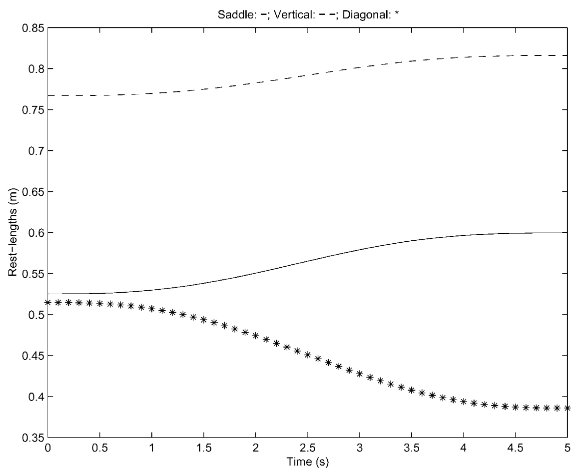

Fig. 5. Rest-lengths time histories for symmetrical reconfiguration.

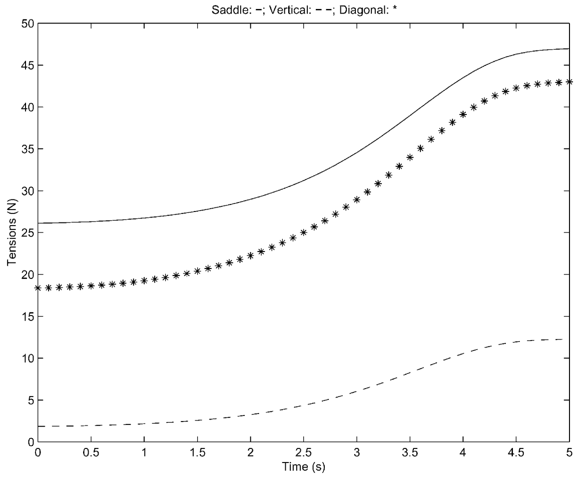

Fig. 6. Tendons tensions time histories for symmetrical reconfiguration.

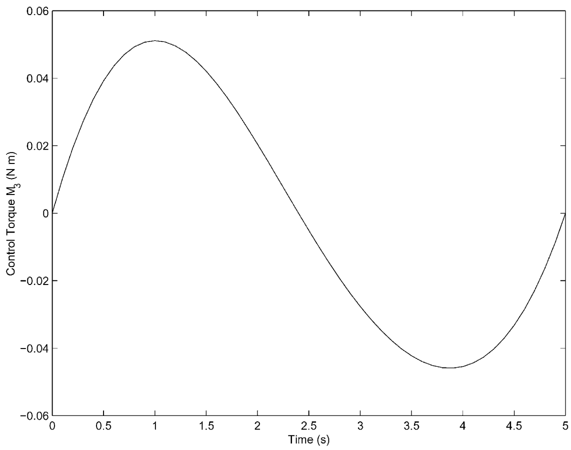

Fig. 7. Control torque time history for symmetrical reconfiguration.

guarantee this trajectory are given by Eqs. (35) and are plotted in Fig. 5. The corresponding tensions time histories, given by Eq. (32) and plotted in Fig. 6, show that throughout the motion all tendons are in tension. The external torque time history, M_3, given by Eq. (33) and plotted in Fig. 7, shows that a control torque is required to perform this symmetrical reconfiguration.

6. Symmetrical reconfiguration with zero control torque

A natural question we ask is if symmetrical reconfigurations in which external torque is not necessary (that is $M_3 = 0$), are possible. In order to answer this question we consider Eq. (22) which, for $M_3 = 0$, reduces to

$$3\tilde{J}\sin^2(\delta)\dot{\alpha} + \frac{\tilde{m}b}{\sqrt{3}}\frac{\mathrm{d}}{\mathrm{d}t}(\sin(\delta)\cos(\alpha+60)) = C_1, \tag{46}$$

where C_1 is a constant. Since the structure is in equilibrium initially, $\dot{\alpha}_i = \dot{\delta}_i = \dot{Z}_i = 0$, thus $C_1 = 0$, and Eq. (46) can be integrated after multiplication with the integrable factor

$$\mu = \frac{1}{\sin^2(\delta)\cos^2(\alpha+60)}. \tag{47}$$

After several algebraic manipulations we obtain from Eq. (46)

$$3\tilde{J}\sin(\alpha-\alpha_i)\sin(\delta)\sin(\delta_i) + \frac{\tilde{m}b}{\sqrt{3}}(\sin(\delta)\cos(\alpha+60) - \sin(\delta_i)\cos(\alpha_i+60)) = 0. \tag{48}$$

Hence the structure can be symmetrically reconfigured such that $M_3 = 0$ if the successive symmetrical configurations the structure passes through during the motion satisfy Eq. (48). Because of this constraint, $\alpha(t)$ and $\delta(t)$ cannot be arbitrarily specified: if one function is specified, then the other is given by Eq. (48). The relations between the time derivatives of $\alpha(t)$ and $\delta(t)$ can be determined by implicit differentiation of Eq. (48). By writing Eq. (48) as $f(\alpha,\delta) = 0$ where

$$f(\alpha,\delta) = 3\tilde{J}\sin(\alpha-\alpha_i)\sin(\delta)\sin(\delta_i) + \frac{\tilde{m}b}{\sqrt{3}}(\sin(\delta)\cos(\alpha+60) - \sin(\delta_i)\cos(\alpha_i+60)), \tag{49}$$

and differentiating $f(\alpha,\delta) = 0$ with respect to time we get:

$$f_\delta\dot{\delta} + f_\alpha\dot{\alpha} = 0, \tag{50}$$

$$f_\delta\ddot{\delta} + f_\alpha\ddot{\alpha} + f_{\alpha\alpha}\dot{\alpha}^2 + 2f_{\alpha\delta}\dot{\alpha}\dot{\delta} + f_{\delta\delta}\dot{\delta}^2 = 0. \tag{51}$$

Here f_α, f_δ, are the first order derivatives of $f(\alpha,\delta)$ with respect to α and δ and $f_{\alpha\delta}$, $f_{\alpha\alpha}$, $f_{\delta\delta}$ are the second order derivatives of $f(\alpha,\delta)$ with respect to α and δ.

For example consider that we specify the function $\alpha(t)$. Then the corresponding $\delta(t)$, which guarantees that $M_3 = 0$, is determined by solving Eq. (48) for $\delta(t)$. Of course there is no guarantee that the corresponding $\delta(t)$ would be of the same class as $\alpha(t)$. However, the implicit function theorem guarantees that if $f_\delta \neq 0$ this is locally true.

6.1. Example

As an example of a symmetrical reconfiguration with zero external torque we consider the same tensegrity structure as before. Consider that a constant force $F_3 = 50$ N acts on the rigid top and that the initial symmetrical equilibrium configuration, at which all tendons are in tension, is characterized by

$$\alpha_i = 70°, \quad \delta_i = 30°, \quad Z_i = 1.6 \text{ m}, \quad T_{r_i} = [0.49\ 8.91\ 8.79]^T \text{ N}. \tag{52}$$

We want to change this configuration to another symmetrical equilibrium configuration characterized by $\alpha_f = 65°$ through a symmetrical reconfiguration with zero external torque. In this case α_f and δ_f must obey Eq. (48) in which α is replaced by α_f and δ is replaced by δ_f. Solving the resulting equation for δ_f we get $\delta_f = 22.03°$. At this point we have to check if in the final configuration characterized by

$$\alpha_f = 65°, \qquad \delta_f = 22.03°, \qquad Z_f = Z_i = 1.6 \text{ m} \tag{53}$$

all tendons are in tension. For this purpose we compute the tensions using Eq. (31) and ascertain that $T_{r_f} = [3.58\ 4.93\ 11.29]^T$ N, showing that, indeed, all tendons are in tension.

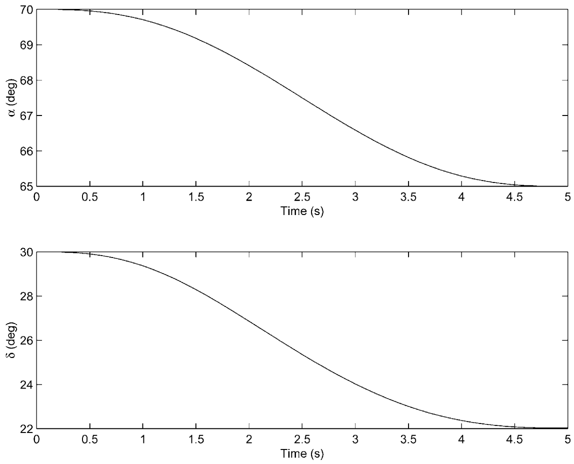

Fig. 8. α and δ time histories for symmetrical reconfiguration with zero control torque.

We shall consider that $\alpha(t)$ is given by Eqs. (38) and (39) and the corresponding $\delta(t)$, which guarantees that $M_3 = 0$, is determined from Eq. (48) (see Fig. 8). The control rest–lengths and tensions required to assure the symmetrical reconfiguration with zero external torque are given in Figs. 9 and 10. We remark that the tendons are always in tension throughout the motion.

Finally we compare this reconfiguration, in which no control torque is required, with the symmetrical reconfiguration in which $\delta(t)$ is no longer given by the condition that $M_3 = 0$ but it is given by Eqs. (40) and (41). The initial and final configurations are the same as before. Fig. 11 shows the control torque, M_3, which has to be applied to the rigid top. We can see that, though of small magnitude, a control torque is necessary in this case.

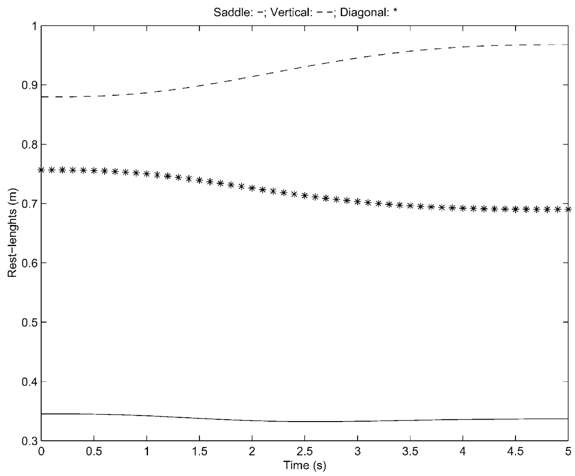

Fig. 9. Rest-lengths time histories for symmetrical reconfiguration with zero control torque.

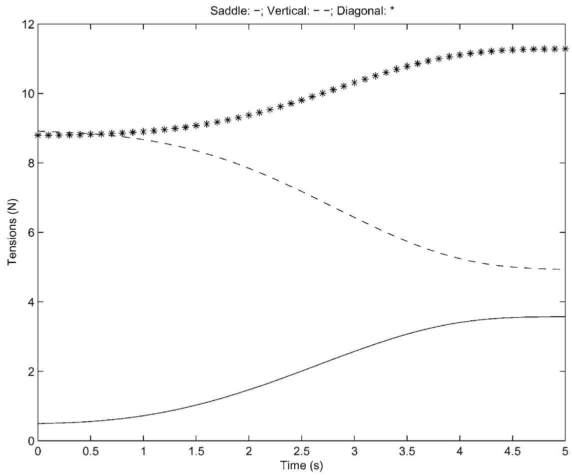

Fig. 10. Tendons tensions time histories for symmetrical reconfiguration with zero control torque.

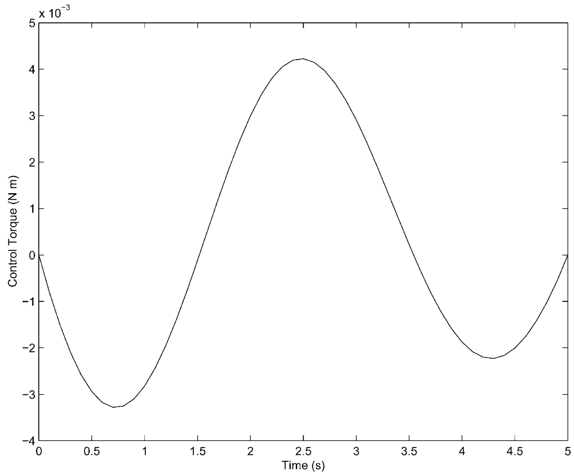

Fig. 11. Control torque time history for symmetrical reconfiguration with α and δ specified.

7. Conclusions

Under very general modeling assumptions the nonlinear equations of motion for tensegrity structures can be derived using Lagrange methodology. This yields a finite set of second order ordinary differential equations. For particular motions of some tensegrity structures, coined symmetrical motions, these equations reduce to a small set. Symmetrical motions can be used for symmetrical reconfigurations of tensegrity structures. An interesting situation in which symmetrical reconfigurations are very useful is when the structure is required to maintain a constant height under a given external force. Tendon control and torque control might be both necessary for this purpose. Through the reconfiguration process the shape as well as the mechanical properties of a tensegrity structure can be modified. The condition under which no control torque is required for symmetrical reconfigurations is derived in the form of an algebraic equation. Numerical examples presented in the article confirm the feasibility of symmetrical reconfigurations.

Appendix A

Matrix A_r is a 3×3 matrix whose elements are given by:

$$A_{r_{11}} = \frac{\sqrt{3}(2Z - 3l\mathrm{c}(\delta))l\mathrm{s}(\delta) - lb\mathrm{c}(\delta)\mathrm{c}(\alpha - 30)}{\sqrt{3}S}, \tag{A.1}$$

$$A_{r_{12}} = -\frac{lb\mathrm{c}(\delta)\mathrm{s}(\alpha + 30)}{V}, \tag{A.2}$$

$$A_{r_{13}} = \frac{\sqrt{3}lZ\mathrm{s}(\delta) - lb\mathrm{c}(\delta)\mathrm{s}(\alpha)}{\sqrt{3}D}, \tag{A.3}$$

$$A_{r_{21}} = \frac{lb\mathrm{s}(\delta)\mathrm{s}(\alpha - 30)}{\sqrt{3}S}, \tag{A.4}$$

$$A_{r_{22}} = -\frac{lb\mathrm{s}(\delta)\mathrm{c}(\alpha + 30)}{V}, \tag{A.5}$$

$$A_{r_{23}} = -\frac{lb\mathrm{s}(\delta)\mathrm{c}(\alpha)}{\sqrt{3}D}, \tag{A.6}$$

$$A_{r_{31}} = \frac{6Z - 12l\mathrm{c}(\delta)}{S}, \tag{A.7}$$

$$A_{r_{32}} = 0, \tag{A.8}$$

$$A_{r_{33}} = \frac{6Z - 6l\mathrm{c}(\delta)}{D}, \tag{A.9}$$

where $\mathrm{c}(*) = \cos(*)$, $\mathrm{s}(*) = \sin(*)$.

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